CHIRAL PERTURBATION THEORY IN THE FRAMEWORK OF NON-COMMUTATIVE GEOMETRY

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Abstract

We consider the non-commutative generalization of the chiral perturbation theory. The resultant coupling constants are severely restricted by the model and in good agreement with the data. When applied to the Skyrme model, our scheme reproduces the non-Skyrme term with the right coefficient. We comment on a similar treatment of the linear σ-model.

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1 Introduction

Discovery of Non-Commutative Geometry (NCG) by Alain Connes [1] and its subsequent application to problems in particle physics has attracted considerable attention in recent years [2, 3, 4, 5, 6, 7]. In its simplest form, the generalized gauge fields of the non-commutative version of the electroweak sector of the standard model naturally included the Higgs field whose origin in the standard model is obscure, to say the least. The mass scale of the Higgs sector is directly connected to the geometrical separation parameter of the NCG gauge theory. Other parameters of the standard model such as mass ratios are also determined in this construction [1]. Subsequently, the non-commutative geometric construction has been applied to the extensions of the standard model [4, 5], to grand unified field theories [8, 9], and even to gravitational interactions [10]. In the former theories, the non-commutative component gives rise to spontaneous symmetry breaking and its Higgs field, while in the latter case a dilaton field emerges from the generalized gravity construction. There is one area however, in which the application of non-commutative framework has not proven straightforward, and that is in QCD. A simple minded procedure will result in an unwanted color symmetry break down [5]. It is entirely possible that more intricate use of non-commutative geometry ideas may tame QCD into its framework, just as a truly non-commutative quantum theory of gravity has eluded us yet, an area in which the primary expectation of non-commutative geometric phenomena lie.

But then there is one region of QCD which lends itself to a similar treatment as in the original application of non-commutative geometry to electroweak sector of standard model. The spontaneous breakdown of the chiral flavour symmetry of QCD in low energies invites a non-commutative exploration. We will take up the application of non-commutative geometric structure to the chiral effective Lagrangian of low energy QCD in this paper.

Chiral perturbation theory (ChPT) as an effective field theory for low energy QCD has been applied successfully to a wide range of problems in hadron physics [11]. In the simplest form ChPT involves the pseudoscalar mesons as the basic fields . Interaction of these fields is described by an effective Lagrangian which is ordered with respect to the number of derivatives and mass of the meson field. To lowest order i.e. second order in momenta,

it simply is the non-linear σ-model which was applied successfully to low energy hadron physics in the early days of current algebra [12]. To fourth order in derivatives, it involves a rather large number of terms, which have enabled a detailed and quantitative analysis of such quantities as form factors and scattering amplitudes for various meson processes [13, 14].

A simple elegant chiral perturbative Lagrangian which involves some quartic terms is the Skyrme model, which has been studied extensively. It describes not only the mesons as the basic fields, but also baryons as the soliton solutions, of the theory.

In its application to hadron physics, it was realized [15] that an extra term, the so called non-Skyrme or symmetric quartic term has to be added, with an adjustable coefficient to describe $\pi\pi$ scattering correctly.

In this paper we will generalize the chiral lagrangian to a two sheeted non-commutative geometry. To second order in momentum, we get the non linear σ-model with a mass term appearing as a result of the non-commutativity of our geometry. To fourth order we obtain the usual chiral perturbation theory lagrangian, with the coupling constants severely restricted. Comparison with the data shows fairly good agreement. In the special region of Skyrme model validity, we obtain both the Skyrme lagrangian and the non-Skyrme term, again with its coefficient determined by the theory and in good agreement with the data.

In section 2, we will review the non-commutative geometry construction and remind the reader of the original application to standard model, thereby setting up our formulation and notation. In section 3, we will apply the non-commutative procedure first to the case of the non-linear σ-model, to the conventional ChPT, and then to the case of the standard Skyrme model. In the appendix we will study the linear σ-model in the framework of non-commutative geometry. [3]

2 Review of Non-Commutative Geometry

In this section we remind some of the relevant features of the non-commutative geometry and set our notation. For detail and further analysis the reader is refered to the references

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[3] After this work was completed we became aware of a preprint [16] which also studied the linear sigma model in the context of non-commutative geometry, but with a different method.

[1, 2, 3]. The basic objects of such a geometry is a possibly non-commutative algebra $\mathcal{A}$ which is the generalization of the algebra of functions on a manifold, and a Dirac k-cycle which is a triplet $(\mathcal{H}, D, \Gamma)$, where $\mathcal{H}$ is a Hilbert space, D is the Dirac operator and Γ is a $\mathbb{Z}_2$ grading operator. The algebra $\mathcal{A}$ is an associative algebra with unit 1 and an involution $*$. A (matrix) representation of $\mathcal{A}$ over the Hilbert space $\mathcal{H}$ is a homomorphism from $\mathcal{A}$ into the linear operators on $\mathcal{H}$ which is faithful. Given an involutive algebra $\mathcal{A}$ which corresponds to a geometrical space, we would like to construct a differential algebra corresponding to a differential geometry. In order to construct this differential algebra we need to define a linear operator d, as the exterior derivative, satisfying $d^2 = 0$ and the Leibniz rule. Using d, we can construct the p-forms as:

$$\sum_i a_0^i \, da_1^i \dots da_p^i \qquad ; a_0^i, a_1^i, \dots, a_p^i \in \mathcal{A}. \tag{1}$$

One defines the representation of an element da in $\mathcal{H}$ as,

$$da = [D, a]_g = Da - \Gamma a \Gamma D. \tag{2}$$

The grading operator Γ satisfies the following properties.

$$\begin{aligned} \Gamma^2 &= 1 \\ \Gamma \omega &= (-)^n \omega \Gamma, \end{aligned} \tag{3}$$

where ω is an n-form in differential algebra. In this formalism the elements of the algebra $\mathcal{A}$ are taken as the 0-forms. The Dirac operator is an unbounded self-adjoint operator in $\mathcal{H}$, such that

$$D\,\Gamma = -\,\Gamma\,D. \tag{4}$$

Other differential geometric quantities such as connection and curvature may be similarly defined [1, 2].

Before applying the above formalism to our physical problem, let us carry out some of calculations which will be needed in the following sections. First we show how the differential algebra reproduces the ordinary differential forms on a flat, compact and Euclidian manifold $\mathcal{M}$. For such a manifold we should take $\mathcal{A}$ to be the algebra of complex valued functions on $\mathcal{M}$,

$$\mathcal{A} := \mathcal{C}^\infty(\mathcal{M}), \tag{5}$$

then D is the ordinary Dirac operator

$$D = \not{\partial} = \gamma^\mu \partial_\mu \tag{6}$$

and γ_5 is the grading operator Γ. According to (2),

$$dg = [D, g], \quad \forall g \in \mathcal{A} \tag{7}$$

which becomes,

$$dg = [\not{\partial}, g] = \gamma^\mu \partial_\mu g \equiv \gamma(\mathbf{d}g), \tag{8}$$

where $\mathbf{d}g$ is the ordinary one-form, $\mathbf{d}g = \partial_\mu g \mathbf{d}x^\mu$.

To describe a fiber bundle over the manifold $\mathcal{M}$, we take

$$\mathcal{A} = \mathcal{C}^\infty(\mathcal{M}) \otimes M_N(\mathcal{C}), \tag{9}$$

where $M_N(\mathcal{C})$ is the algebra of $N \times N$ complex matrices. In this case the Dirac and grading operators are $\not{\partial} \otimes 1$ and $\gamma_5 \otimes 1$ respectively. Next we construct the differential algebra of the fiber bundle over the space consisting of two layers, each layer a Euclidian, compact manifold $\mathcal{M}$. Each layer is described by the algebra of continuous functions $\mathcal{C}^\infty(\mathcal{M})$. The proper algebra for this geometry is:

$$\mathcal{A} = \mathcal{C}^\infty(\mathcal{M}) \otimes (\ M_N(\mathcal{C}) \oplus M_N(\mathcal{C})\ ), \tag{10}$$

and the Dirac operator is [8]

$$D = \begin{pmatrix} \not{\partial} \otimes \mathbf{1} & \gamma_5 \otimes M \\ \gamma_5 \otimes M^\dagger & \not{\partial} \otimes \mathbf{1} \end{pmatrix} \tag{11}$$

Setting the off-diagonal terms equal to zero, will make differentiation on one layer independent of the other layer.

In this example we take the representation of an elements of $\mathcal{A}$ to be

$$g = \begin{pmatrix} V(x) & 0 \\ 0 & V'(x) \end{pmatrix}, \quad g \in \mathcal{A} \tag{12}$$

where $V(x)$ and $V'(x)$ are $N \times N$ matrices and x indicates the coordinates on $\mathcal{M}$. This representation contains in itself the information about the two layers of space. $V(x)$ represents functions on one layer and $V'(x)$ those of the other layer.

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Now we can calculate dg. With the same procedure which was used for obtaining (8), we find

$$dg \ = \ [\,D\,,\,g\,] = \begin{pmatrix} \partial\!\!\!/ V & \gamma_5(M\,V' - V\,M) \\ \gamma_5(M^\dagger\,V - V'\,M^\dagger) & \partial\!\!\!/ V' \end{pmatrix}. \tag{13}$$

In the above formula not only we have $\partial\!\!\!/ V$ and $\partial\!\!\!/ V'$ which can be related to the ordinary one-forms on each layer of space, but also there are off-diagonal terms which are proportional to the difference of the functions corresponding to the two seperate layers. In fact the $N \times N$ matrix M which is called the mass matrix, establishes the connection between the two layers.

A two sheeted space also was used by Connes [1, 2] to reproduce the Higgs sector of the standard model beside the usual gauge boson sector. For this purpose, notice that the curvature two form $\theta = d\rho + \rho^2$ corresponding to the connection one-form $\rho = \sum_i a_i\, db_i$, $\forall\, a_i\,,\, b_i \in \mathcal{A}$ with, $\rho = \rho^*$, yields the Yang-Mills action

$$S_{YM} = \frac{1}{4}Tr_\omega(\theta^2 \mid D \mid^{-4}), \tag{14}$$

where Tr_ω is the Dixmier trace and $\mid D \mid^2 = DD^\dagger$. It can be shown in this case that the S_{YM} will reduce to

$$S_{YM} = \frac{1}{4}\int dx^4 \sqrt{g}Tr(tr(\theta^2)), \tag{15}$$

where tr is taken over the Clifford algebra and Tr is taken over the matrix structure [8].

The fermionic part of the action is,

$$S_F = \int dx^4 \overline{\Psi}(D + \rho)\Psi. \tag{16}$$

and the total action $S = S_F + S_{YM}$ can easily be shown to be invariant under the gauge transformations, $\Psi \to \Psi' = g\Psi$ and $\rho \to g\rho g^* + gdg^*$, with g an element of the unitary group $\mathcal{U}$ of $\mathcal{A}$

$$g \ \in \mathcal{U}(\mathcal{A}) = \{g \in \mathcal{A}\ , gg^* = g^*g = 1\}, \tag{17}$$

and $*$ indicating involution in $\mathcal{A}$.

To obtain the standard model electroweak lagrangian Connes takes the algebra to be

$$\mathcal{A} = C^\infty(\mathcal{M}) \otimes (\mathbb{C} \oplus \mathbb{H}) \tag{18}$$

where $\mathbb{C}$ and $\boldsymbol{H}$, denote the algebra of complex numbers and quaternions. Using a Dirac operator similar to (11), the connection one form is obtained via eq.(7) [8].

$$\rho = \begin{pmatrix} A_1 & \gamma_5 H K \\ \gamma_5 H^* K^* & A_2 \end{pmatrix} \qquad (19)$$

Here A_1 and A_2 are the ordinary gauge fields over each layer of space and H is the other component of the gauge field along the discrete dimension which acts as the Higgs field in the theory. K is a mixing matrix related to the fermionic mass matrix. The complete action of the electroweak sector of the standard model is then obtained from the generalized gauge field [4].

3   Chiral perturbation theory and NCG

Chiral perturbation theory is an effective field theory of mesons, for low energy QCD. The strategy in this theory is to expand the effective lagrangian in powers of the momenta and take the lowest order terms. To obtain the effective lagrangian, one demands that the symmetry of this effective theory be a symmetry of QCD. As a good approximation QCD has $SU_L(N_f) \times SU_R(N_f)$ chiral symmetry where N_f is the number of massles flavours. On the other hand Lorentz invariance forces the number of derivatives in each term of the effective lagrangian expansion to be even. Another important property of effective Lagrangian comes from PCAC, which at low energies prevents the Goldstone bosons interact with one another. Putting all these together one may write the effective lagrangian of low energy QCD as follows [17] :

$$\mathcal{L}_{eff} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + ... \qquad (20)$$

$\mathcal{L}^{(2)}$ has the form of the 4 dimensional non-linear σ-model, i.e.

$$\mathcal{L}^{(2)} = \frac{F_0^2}{4} Tr(\partial_\mu U \partial^\mu U^\dagger) \ ; \ L_\mu = U \partial_\mu U^\dagger \, , \, U \in SU(N_f), \qquad (21)$$

where U is related to the mesonic field by

$$U = e^{\frac{i\tau^a \pi_a}{F_0}} \qquad (22)$$

and F_0 is the pion decay constant. If pions are taken to be massive, then it is not difficult to show that $\mathcal{L}^{(2)}$ should be modified as bellow [17, 18]

$$\mathcal{L}^{(2)} = \frac{F_0^2}{4}Tr(L_\mu L^\mu) + \frac{F_0^2 m_\pi^2}{4}Tr(U + U^\dagger - 2). \tag{23}$$

Higher order terms of (20) can be written in terms of U or L_μ, and increase in complexity as the order increases. Gasser and Leutwyler [14] have obtained the following expression for the general Lagrangian to order p^4 in the case of $N_f = 3$:

$$
\begin{aligned}
\mathcal{L}^{(4)} &= L_1(Tr(\partial^\mu U \partial_\mu U^\dagger))^2 + L_2 Tr(\partial_\mu U \partial_\nu U^\dagger) Tr(\partial^\mu U \partial^\nu U^\dagger) \\
&+ L_3 Tr(\partial^\mu U \partial_\mu U^\dagger \partial^\nu U \partial_\nu U^\dagger) + L_4 Tr(\partial^\mu U \partial_\mu U^\dagger) Tr(\chi^\dagger U + \chi U^\dagger) \\
&+ L_5 Tr(\partial^\mu U \partial_\mu U^\dagger (\chi^\dagger U + U^\dagger \chi)) + L_6[Tr(\chi^\dagger U + \chi U^\dagger)]^2 \\
&+ L_7(Tr(\chi^\dagger U - \chi U^\dagger))^2 + L_8 Tr(\chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger)
\end{aligned}
\tag{24}
$$

In the above lagrangian we have ignored the presence of vector and axial fields which occur in the original lagrangian and only kept the symmetry breaking terms. The field χ contains the information about the mesonic masses, which to lowest order is the mass matrix $\chi = m_\pi \mathbb{1}$. Then the lagrangian becomes,

$$
\begin{aligned}
\mathcal{L}^{(4)} &= L_1(Tr(\partial^\mu U \partial_\mu U^\dagger))^2 + L_2 Tr(\partial_\mu U \partial_\nu U^\dagger) Tr(\partial^\mu U \partial^\nu U^\dagger) \\
&+ L_3 Tr(\partial^\mu U \partial_\mu U^\dagger \partial^\nu U \partial_\nu U^\dagger) + L_4 m_\pi^2 Tr(\partial^\mu U \partial_\mu U^\dagger) Tr(U + U^\dagger) \\
&+ L_5 m_\pi^2 Tr(\partial^\mu U \partial_\mu U^\dagger (U + U^\dagger)) + L_6 m_\pi^4 [Tr(U + U^\dagger)]^2 \\
&+ L_7 m_\pi^4 (Tr(U - U^\dagger))^2 + L_8 m_\pi^4 Tr(U^2 + U^{\dagger 2})
\end{aligned}
\tag{25}
$$

L_i are low energy coupling constants which in principle can be determined from QCD. These coupling constants are obtained by comparison with experiments such as $\pi\pi$ scattering . Recently Fearing [19] has obtained an expression for $\mathcal{L}^{(6)}$ which has some hundred terms.

Many years ago before a systematic study of chiral perturbation theory as an effective theory for low energy QCD was embarked upon, Skyrme [20] proposed the following Lagrangian as an effective theory of hadrons,

$$\mathcal{L}_{Sk} = \frac{-F_0^2}{4}Tr(L_\mu L^\mu) + \frac{1}{32e^2}Tr[L_\mu, L_\nu]^2 \tag{26}$$

The first term in (26) is the well known non-linear σ-model term eq.(21), and the second term which is called Skyrme term is responsible for the soliton solutions of the theory.

But, if one expands the Skyrme Lagrangian (26) in terms of pion fields it can be seen that such interactions as $\pi^0\pi^0 \to \pi^0\pi^0$ are forbidden [21, 22, 23]. Now in the limit $m_\pi \to 0$, there are only two possible independent quartic derivative terms. One of them is the Skyrme term above, and the other is $\frac{\gamma}{8e^2}Tr(L_\mu L^\mu)^2$ [24]. By adding this term to the Skyrme Lagrangian,

$$\mathcal{L}_{mod.Sk} = \frac{-F_0^2}{4}Tr(L_\mu L^\mu) + \frac{1}{32e^2}Tr[L_\mu, L_\nu]^2 + \frac{\gamma}{8e^2}Tr(L_\mu L^\mu)^2 \,, \tag{27}$$

not only the interaction $\pi^0\pi^0 \to \pi^0\pi^0$ is now included but also the accuracy of the quantitative results improve [15, 21]. Note that the coupling constants in the Lagrangian of the extended Skyrme model (27) are related to the coupling constants L_i in (25).

We are now in the position to develop the chiral perturbation theory in the framework of non-commutative geometry. As mentioned in section 2, the basic tools for model building in the framework of non-commutative geometry are two things. First a suitable algebra $\mathcal{A}$ which describes our geometrical space and second a k-cycle $(\mathcal{H}, D, \Gamma)$ which helps us to develop a differential calculus on $\mathcal{A}$.

In choosing $\mathcal{A}$, we must be guided by the form of the one-form L_μ which appear in the ordinary ChPT,

$$L = L_\mu \mathbf{d}x^\mu = (U\partial_\mu U^\dagger)\mathbf{d}x^\mu \quad , \quad UU^\dagger = U^\dagger U = \mathbf{1}. \tag{28}$$

and compare it with (8). It is then obvious that $\gamma(L)$ in non-commutative version corresponds to L, for a single layer space,

$$\gamma(L) = \gamma^\mu U\partial_\mu U^\dagger = U[\not\partial, U^\dagger] = UdU^\dagger \quad , U \in \mathcal{U}(\mathcal{A}) \tag{29}$$

where d is the exterior derivative on differential algebra. We should therefore identify,

$$L = gdg^* \,, \tag{30}$$

with g in a representation of $\mathcal{A}$.

Comparison with eq.(20) - (25) suggests that the simplest generalization of the effective lagrangian in non-commutative geometry up to order p^4 should read:

$$\begin{aligned}
\mathcal{L}_{eff} &= K_1 Tr(gdg^*gdg^*) + K_2 Tr(gdg^*gdg^*gdg^*gdg^*) \\
&+ K_3 (Tr(gdg^*gdg^*))^2
\end{aligned} \tag{31}$$

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where K_i are the coupling constants of theory similar to L_i (24) or (25). It is interesting to note that the first and third order terms in powers of momenta, vanish due to vanishing of the trace of odd number of Dirac matrices.

It is instructive to apply our non-commutative geometry machinery to the ordinary four dimensional manifold. For this purpose, we take $\mathcal{A}$ to be as in eq.(9), and consider the second order terms of the lagrangian (31) only, then with $g = U(x) \in SU(N_f)$, we get,

$$
\begin{aligned}
\mathcal{L}_{eff}^{(2)} &= K_1 Tr(gdg^* \, gdg^*) = (-K_1)Tr(dg \, dg^*) \\
&= (-K_1)Tr([D,g][D,g^*]) = (-K_1)Tr(\gamma^\mu \gamma^\nu)Tr(\partial_\mu U \partial_\nu U^\dagger) \\
&= (-4K_1)Tr(\partial_\mu U \partial^\mu U^\dagger)
\end{aligned}
\tag{32}
$$

This result is nothing but the 4-dimensional non-linear σ-model as we expected. By comparison of the lagrangian in (32) and (21),

$$
K_1 = -\frac{F_0^2}{16}.
\tag{33}
$$

Again before applying our method to the more general lagrangian of eq.(31) let us confine ourselves to the first term still, but use the two layer geometry of eqs.(10) and (11), with N the number of flavours and

$$
g = \begin{pmatrix} U & 0 \\ 0 & U' \end{pmatrix} \quad , \quad U, U' \in U(N)
\tag{34}
$$

Repeating the calculation in (32) we get,

$$
\begin{aligned}
\mathcal{L}_{eff}^{(2)} &= (-4K_1)Tr[\partial_\mu U \partial^\mu U^\dagger + (MU' - UM)(M^\dagger U^\dagger - U'^\dagger M^\dagger) \\
&+ \partial_\mu U'^\dagger \partial^\mu U'^\dagger + (M^\dagger U - U'M^\dagger)(MU'^\dagger - U^\dagger M)].
\end{aligned}
\tag{35}
$$

For $M = 0$ we simply get two independent non-linear σ-models on two separate layers. In general, if we assume only

$$
MM^\dagger = M^\dagger M = m^2 \mathbf{1},
\tag{36}
$$

and set $U' = \mathbf{1}$, for simplicity, we get

$$
\mathcal{L}_{eff}^{(2)} = (-4K_1)Tr(\partial_\mu U \partial^\mu U^\dagger) + (-2K_1)m^2 Tr(U + U^\dagger - 2),
\tag{37}
$$

which is the lagrangian of eq.(23), with $\frac{F_0^2 m_\pi^2}{4} Tr(U + U^\dagger - 2)$ the symmetry breaking term. Thus we have generated the pion mass naturally. According to Connes's interpretation [2, 3], the distance between the layers of space is proportional to $\frac{1}{m}$. So if we let the distance of these two layers tend to infinity, the non linear σ-model is reproduced.

We will now take up the lagrangian to order 4 on the two layer space above,

$$
\begin{aligned}
\mathcal{L}_{eff}^{(4)} &= K_2 Tr(gdg^* gdg^* gdg^* gdg^*) + K_3 (Tr(gdg^* gdg^*))^2 \\
&= K_2 Tr(dgdg^* dgdg^*) + K_3 (Tr(dgdg^*))^2.
\end{aligned}
\tag{38}
$$

assuming $U' = 1$ again and eq.(36), a lengthy calculation leads to,

$$
\begin{aligned}
\mathcal{L}_{eff} &= -(4K_1 + 32K_2 m^2 - 96K_3 m^2) Tr(\partial_\mu U \partial^\mu U^\dagger) \\
&- (2K_1 + (64K_2 + 16K_3) m^4) Tr(U + U^\dagger) + (-2K_2 + 16K_3)(Tr(\partial_\mu U \partial^\mu U^\dagger)^2 \\
&- 4K_2 Tr(\partial^\mu U \partial^\nu U^\dagger) Tr(\partial_\mu U \partial_\nu U^\dagger) + 16K_2 m^2 Tr(\partial^\mu U \partial_\mu U^\dagger \partial^\nu \partial_\nu U^\dagger) \\
&+ 16K_3 m^2 Tr(\partial_\mu U \partial^\mu U^\dagger) Tr(U + U^\dagger) + 16K_2 m^2 Tr(\partial^\mu U \partial_\mu U^\dagger (U + U^\dagger)) \\
&+ 4K_3 m^4 (Tr(U + U^\dagger))^2 + 8K_2 m^4 Tr(U^2 + U^{\dagger 2}) + \text{constant}
\end{aligned}
\tag{39}
$$

Aside from the quadratic terms already recovered at the level of p^2, we have therefore obtained all the terms in the ordinary ChPT to order p^4, except for the L_7 term. By comparing (39) with (25) and (23) we may write the following relations between the parameters of (39) and physical parameters L_i, F_0 and m_π,

$$
\frac{-F_0}{4} = 4K_1 + 32K_2 m^2 + 96K_3 m^2
\tag{40}
$$

$$
\frac{-F_0}{4} m_\pi^2 = 2K_1 m^2 + 64K_2 m^4 + 16K_3 m^4
\tag{41}
$$

$$
L_1 = -2K_2 + 16K_3
\tag{42}
$$

$$
L_2 = -4K_2
\tag{43}
$$

$$
L_3 = 16K_2
\tag{44}
$$

$$
L_4 = 16K_3 \left(\frac{m^2}{m_\pi^2} \right)
\tag{45}
$$

$$
L_5 = 16K_2 \left(\frac{m^2}{m_\pi^2} \right)
\tag{46}
$$

$$
L_6 = 4K_3 \left(\frac{m^4}{m_\pi^4} \right)
\tag{47}
$$

$$
L_8 = 8K_2 \left(\frac{m^4}{m_\pi^4} \right)
\tag{48}
$$

10

It is to be noted that we have reproduced a whole series of terms in the ChPT lagrangian, just starting from the simple form (31). At the ρ meson mass, $M_\rho = 770$ Mev, and decay constant $F_0 = 154$ Mev, we take $L_3 = (-4.4 \pm 2.5) \times 10^{-3}$, and $L_1 = (0.7 \pm 0.3) \times 10^{-3}$ as inputs from [14, 25] and obtain the corresponding values for the remaining parameters,

$$
\begin{aligned}
L_2 &= (1.1 \pm 0.6) \times 10^{-3}, \, [\, (1.3 \pm 0.7) \times 10^{-3}\,] \\
L_4 &= (0.1 \pm 0.5) \times 10^{-3}, \, [\, (-0.3 \pm 0.5) \times 10^{-3}\,] \\
L_5 &= (-3.1 \pm 2.4) \times 10^{-3}, \, [\, (1.4 \pm 0.5) \times 10^{-3}\,] \\
L_6 &= (0.02 \pm 0.08) \times 10^{-3}, \, [\, (-0.2 \pm 0.3) \times 10^{-3}\,] \\
L_7 &= 0, \, [\, (0.4 \pm 0.15) \times 10^{-3}\,] \\
L_8 &= (-1.1 \pm 1.1) \times 10^{-3}, \, [\, (0.9 \pm 0.3) \times 10^{-3}\,]
\end{aligned}
\tag{49}
$$

For the purpose of comparison we have also denoted the experimental values in the brackets, taken from ref. [14, 25]. The agreement for L_2, L_4, L_6 are good, while L_5, L_7, and L_8 are in disagreement.

Had we limited ourselves to the first two terms of our non-commutative lagrangian eq. (31), i.e. $K_3 = 0$, and set the mass scale $m = 0$, we would have gotten the Skyrme model lagrangian, together with the non Skyrme term of eq.(27),

$$
\mathcal{L}^{(4)} = -2K_2 Tr([L_\mu, L_\nu]^2) + 4K_2 Tr(L_\mu L^\mu)^2,
\tag{50}
$$

which gives,

$$
K_2 = -\frac{1}{64e^2}
\tag{51}
$$

$$
\gamma = -\frac{1}{2}.
\tag{52}
$$

This value of γ agrees within the data at 1 Gev energy within the experimental error. Surprisingly, it also agrees with the non-Skyrme term found by Anderianov [26] from bosonization method in QCD within a minus sign[4] (see also [27]).

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[4]We would like to thank Maxim Polyakov for clarification of this point.

Acknowledgements

Appendix

An important feature of the linear σ-model [28] is its built in mechanism of spontaneous symmetry breaking, which may indicate use of the formalism of non-commutative geometry; and we will embark upon in this appendix. Recently Guo, et al [16], have used another formalism of non-commutative geometry, [29], and constructed the linear σ-model in that framework.

As in section 3, let us take the algebra $\mathcal{A}$ and Dirac operator D as (10) and (11) respectively, with $N = 2$; then the one-form ρ is,

$$\rho = \sum_i a^i \, [\, D \, , \, b^i \,] \qquad , a^i \, , \, b^i \in \mathcal{A} \tag{A.1}$$

where a^i and b^i are represented by,

$$a^i \rightarrow \ \text{diag} \ (a^i_1, a^i_2) \quad , \quad b^i \rightarrow \ \text{diag} \ (b^i_1, b^i_2) \tag{A.2}$$

and a^i_j and b^i_j are 2×2 matrices. Then by a straightforward calculation it is seen that,

$$\rho = \begin{pmatrix} A_1 & \gamma_5 \otimes \phi_{12} \\ \gamma_5 \otimes \phi_{21} & A_2 \end{pmatrix} \tag{A.3}$$

where

$$
\begin{aligned}
A_m &= \sum_i a^i_m \, \slashed{\partial} b^i_m, \quad m = 1, 2, \\
\phi_{mn} &= \sum_i a^i_m (M b^i_n - b^i_m M), \quad m \neq n,
\end{aligned}
\tag{A.4}
$$

By condition $\rho = \rho^*$, A_m and ϕ_{mn}, satisfy

$$A^*_m = A_m \quad \text{and} \quad \phi^*_{mn} = \phi_{mn} \tag{A.5}$$

To find the curvature θ, let us first calculate

$$d\rho = \sum_i da^i \, db^i = \sum_i [\, D \, , \, a^i \,] \, [\, D \, , \, b^i \,]. \tag{A.6}$$

Matrix elements of $d\rho$ are:

$$
\begin{aligned}
(d\rho)_{11} &= \slashed{\partial} A_1 + (M \phi_{21} + \phi_{12} M^\dagger) - X_1 \\
(d\rho)_{12} &= -\gamma_5 (\gamma^\mu \partial_\mu \phi_{12} + A_1 M - M A_2) \\
(d\rho)_{21} &= -\gamma_5 (\gamma^\mu \partial_\mu \phi_{21} + A_1 M - M A_2) \\
(d\rho)_{22} &= \slashed{\partial} A_2 + (M^\dagger \phi_{21} + \phi_{12} M) - X_2
\end{aligned}
\tag{A.7}
$$

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where X_1 and X_2 are the auxiliary fields

$$X_m = \sum_i \gamma^\mu \gamma^\nu a_m^i \partial_\mu \partial_\nu b_m^i + a_m^i [MM^\dagger, b_m^i], \quad m = 1, 2, \tag{A.8}$$

Using the standard methods [8] we eliminate these auxiliary fields and obtain the form of θ in terms of the gauge fields. We then use the Dirac operator,

$$D = \begin{pmatrix} \partial\!\!\!/ \otimes \mathbb{1} \otimes \mathbb{1} & \gamma_5 \otimes M \otimes K \\ \gamma_5 \otimes M^\dagger \otimes K^\dagger & \partial\!\!\!/ \otimes \mathbb{1} \otimes \mathbb{1} \end{pmatrix} \tag{A.9}$$

and set $A_{i\mu} = 0$ to restrict ourselves to the Higgs field

$$\theta = \begin{pmatrix} \mathbb{1} \otimes (HH^\dagger + m^2) \otimes KK^\dagger & -\gamma^5 \gamma^\mu \otimes \partial_\mu H \otimes K \\ -\gamma^5 \gamma^\mu \otimes \partial_\mu H^\dagger \otimes K^\dagger & \mathbb{1} \otimes (H^\dagger H + m^2) \otimes K^\dagger K \end{pmatrix} \tag{A.10}$$

where

$$H = \phi + M \ \in M_2(\mathbb{C}). \tag{A.11}$$

One may expand H in terms of the Pauli matrices τ

$$H = \sigma' \mathbb{1} + i\pi'.\tau \tag{A.12}$$

where π' is a three component (π'_1, π'_2, π'_3) object. Then for Yang-Mills action we obtain

$$S_{YM} = \int dx^4 \{ -8Tr(KK^\dagger)[(\partial_\mu \sigma')^2 + (\partial_\mu \pi')^2] + 8Tr(KK^\dagger)^2(\sigma'^2 + \pi'^2 + m^2)^2 \} \tag{A.13}$$

For the fermionic part of theory first we construct the operator $D + \rho$,

$$D + \rho = \begin{pmatrix} \partial\!\!\!/ \otimes \mathbb{1} \otimes \mathbb{1} & \gamma^5 \otimes H \otimes K \\ \gamma^5 \otimes H^\dagger \otimes K^\dagger & \partial\!\!\!/ \otimes \mathbb{1} \otimes \mathbb{1} \end{pmatrix} \tag{A.14}$$

then taking the fermionic field Ψ as a two component field,

$$\Psi = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} \tag{A.15}$$

where Ψ_L and Ψ_R are the left and right handed spinor fields, one can show that,

$$S_F = \int d^4x \{ \bar{\Psi}_L \partial\!\!\!/ \Psi_L + \bar{\Psi}_R \partial\!\!\!/ \Psi_R + \bar{\Psi}_L (\sigma' + i\tau.\pi') \Psi_R K + \bar{\Psi}_R (\sigma' + i\tau.\pi') \Psi_L K^\dagger \}. \tag{A.16}$$

Finally adding S_{YM} and S_F, the total action is

$$S = \int d^4x \{ \bar{\Psi}_L \partial\!\!\!/ \Psi_L + \bar{\Psi}_R \partial\!\!\!/ \Psi_R + \bar{\Psi}_L (\sigma + i\tau.\pi) \Psi_R \frac{K}{\alpha} + \bar{\Psi}_R (\sigma + i\tau.\pi) \Psi_L \frac{K^\dagger}{\alpha}$$
$$+ 1/2[(\partial_\mu \sigma)^2 + (\partial_\mu \pi)^2] - \frac{\beta^2}{4\alpha^2}(\sigma^2 + \pi^2 + m^2)^2 \} \tag{A.17}$$

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where

$$- 8\, Tr(KK^\dagger) = \frac{1}{2}\alpha^2 \ , \ \ 8\, Tr(KK^\dagger)^2 = -\frac{1}{4}\beta^2 \ , \ \ \sigma = \alpha\sigma' \ , \ \ \pi = \alpha\pi' \qquad (A.18)$$

If we compare the above action with the standard action of the linear sigma model [30]

$$S_{L\sigma} = \int d^4x \{ \bar{\Psi}_L \not{\partial}\Psi_L + \bar{\Psi}_R \not{\partial}\Psi_R + g\, \bar{\Psi}_L(\sigma + i\tau.\pi)\Psi_R + g\, \bar{\Psi}_R(\sigma + i\tau.\pi)\Psi_L$$

$$+ 1/2[(\partial_\mu\sigma)^2 + (\partial_\mu\pi)^2] - \frac{\lambda}{4}(\sigma^2 + \pi^2 + \frac{\mu^2}{\lambda})^2 \} \qquad (A.19)$$

the parameters in the lagrangian (A.17) are

$$g = \frac{K}{\alpha} \ , \ \ \lambda = \frac{\beta^2}{\alpha^2} \ , \ \ \frac{\mu^2}{\lambda} = \alpha^2 m^2. \qquad (A.20)$$

For pion decay constant and the nucleon mass we obtain:

$$F_0 = \mid \alpha^2 m^2 \mid^{\frac{1}{2}} \ \ \text{and} \ \ M_N = \frac{F_0}{4} \qquad (A.21)$$

reminiscent of the Goldberger-Treiman relation.

References

[1] A. Connes, *Non-Commutative Geometry*, Publ. Math. IHES **62** (1985);
    A. Connes, *Non-Commutative Geometry*, IHES/M/93/12 (1993).

[2] A. Connes and J. Lott, Nucl. Phys. Proc. Suppl. **B18**, 29 (1989).

[3] A. Connes, in *The Interface of Mthematics and Particle Physics*, eds. D. Quillen et al. (Clarendon, Oxford,1990).

[4] D. Kastler, Rev. Math. Phys. **5**, 477 (1993).

[5] D. Kastler and T. Schucker, Theor. Math. Phys. **92**, 522 (1992).

[6] B. S. Balakrishna, F. Gursey and K. E. Wail, Phys. Lett. **B254**, 430 (1991).

[7] T. Shucker, J. M. Zylinski, *Connes Model Building Kit*, preprint CPt-93/p.2960 (1993).

[8] A. Chamseddine, G. Felder and J. Frohlich, Phys. Lett. **B296**, 109 (1993), Nucl. Phys. **B395**, 672 (1993).

[9] A. Chamseddine, J. Frohlich, *SO(10) Unification in Non-Commutative Geometry*, preprint ZU-TH-10/1993, ETH/TH/93-12 (1993).

[10] A. Chamseddine, G. Felder and J. Frohlich, *Gravity in Non-Commutative Geometry* , preprint ETH-TH/1992-18 (1992).

[11] M. Dey and J. Dey, *Nuclear and Particle Physics*, (Springer-Verlag, 1994).

[12] B. W. Lee and H. T. Nieh, Phys. Rev. **166**, 1507 (1968);
    S. Gasiorowicz and D. A. Grefen, Rev. Mod. Phys. **44**, 531 (1969).

[13] J. Gasser and H. Leutwyler, Ann. Phys. **158**, 142 (1984).

[14] J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 465 (1985).

[15] J. F. Donoghue , E. Golowich and B. R. Holstein , Phys. Rev. Lett. **53**, 747 (1984).

[16] H. Y. Guo, J. M. Li and K. Wu, *The σ-Model and Non-Commutative Geometry*, preprint hep-ph/9408150 (1994).

[17] H. Leutwyler , *Principles of Chiral Perturbation Theory*, preprint BUTP-94/13 (1994).

[18] G. Holzwarth and B. Schwesinger , Rep. Prog. Phys. **49**, 825 (1986).

[19] H. W. Fearing and S. Scherer, *Extension of the Chiral Perturbation Theory Meson Lagrangian to Order p^6*, preprint hep-ph/94008346 (1994).

[20] T. H. R. Skyrme, Proc. R. Soc. **A260**, 127 (1961).

[21] T. N. Pham and T. N. Truong, Phys. Rev. **D31**, 3027 (1984).

[22] T. N. Truong, in *Workshop on Skyrmions and Anomalies*, (World Scientific, Singapor, 1987), pp. 101-111.

[23] R. K. Bhaduri, *Models of the Nucleon*, (Addison-Wesley, 1988).

[24] J. Gasser and H. Leutwyler, Phys. Rev. Lett. **125B**, 321 and 325 (1983).

[25] G. Ecker , J. Gasser ,A. Pich and E. De Rafael, Nucl. Phys. **B321**, 311 (1989).

[26] A. A. Andrianov Phys. Lett. **B157**, 425 (1985).

[27] V. G. Makhakov, V. P. Rybakov and V. I. Sanyuk,*The Skyrme Model* (Springer-Verlag, 1993).

[28] M. Gell-Mann and M. Levy, Nuovo Cimento **16**, 705 (1960).

[29] A.Sitarz, *Non-Commutative Geometry and Gauge Theory on Discrete Groups*, preprint TPJU-7/1992 (1992).

[30] S. Pokorski, *Gauge Field Theories*,(Cambridge Univ. Press, 1987).